RESTRICTION ESTIMATES FOR THE SPACE CURVES WITH RESPECT TO GENERAL MEASURES

SEHEON HAM AND SANGHYUK LEE

Abstract. In this paper we consider restriction estimates for the space curves with respect to general measures and obtain the optimal decay estimates when the curve satisfies a finite type condition. The argument here is new in that it doesn't rely on the *offspring curve* method which has been previously used. Our work was inspired by the recent argument due to Bourgain and Guth which was used to deduce linear restriction estimate from multilinear estimate for hypersurfaces.

1. introduction

Let $\gamma : I = [0,1] \to \mathbb{R}^d$, $d \ge 2$ be a smooth function. For $\lambda \ge 1$ we define an oscillatory integral operator by

$$T_\lambda^\gamma f(x) = \int_I e^{i\lambda x\cdot\gamma(t)} f(t)dt.$$

This operator is a dual form of the Fourier restriction to the curve $\lambda\gamma(t)$, $t \in I$. Let ν be a measure in $\mathbb{R}^d$ and $1 \le p, q \le \infty$. We consider the oscillatory estimate

$$\|T_\lambda^\gamma f\|_{L^q(d\nu)} \le C\lambda^{-\beta}\|f\|_{L^p(I)}. \tag{1}$$

Nondegenerate curves. It is well known that the range of p, q is related to the curvature condition of γ. When ν is the Lebesgue measure the problem of obtaining the estimate (1) has been considered by various authors [28, 24, 11, 16] (also see [18, 19, 7, 6, 8, 15]). Under the assumption

$$\det(\gamma'(t), \gamma''(t), \cdots, \gamma^{(d)}(t)) \ne 0 \tag{2}$$

for all $t \in I$ which we call the nondegenerate condition, it is known that (1) holds with $\beta = d/q$ if

$$\frac{d(d+1)}{2q} + \frac{1}{p} \le 1 \quad \text{and} \quad q > \frac{d^2+d+2}{2}. \tag{3}$$

In two dimension this is due to Zygmund [28] and a generalization to oscillatory integral was obtained by Hörmander [22] (see [20] for earlier work by Fefferman and Stein). In higher dimensions $d \ge 3$ the estimates on the full range were proved by Drury [16] after earlier partial results by Prestini [24] and Christ [11]. The necessity of $d(d+1)/2q + 1/p \le 1$ can be shown by a Knapp type example. When $\gamma(t) = (t, t^2, \cdots, t^d)$ and $d \ge 3$, by a result due to Arkhipov, Chubarikov and Karatsuba

2010 *Mathematics Subject Classification.* 42B10.
Key words and phrases. Restriction estimate, space curves, affine arclength measure.
Supported in part by NRF grant 2012008373 (Republic of Korea).

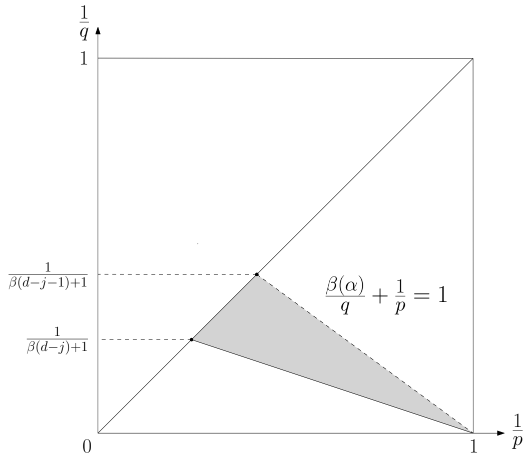

FIGURE 1. For $d-j-1<\alpha\le d-j$, the edge line $\beta(\alpha)/q+1/p=1$ is contained in the marked area. The region given by $\beta(\alpha)/q+1/p<1$ gets larger as α decreases.

[1] it follows that the condition $q>(d^2+d+2)/2$ is necessary. The operator T_λ^γ can also be generalized by replacing $x\cdot\gamma(t)$ with $\phi(x,t)$. In this case, Bak and the second author [3] showed that (1) holds with $\beta=d/q$ for p,q satisfying (3) whenever $\det(\partial_t(\nabla_x\phi),\partial_t^2(\nabla_x\phi),\cdots,\partial_t^d(\nabla_x\phi))\neq 0$ holds. Bak, Oberlin and Seeger [7] showed a weak type estimate for the critical $p=q=(d^2+d+2)/2$.

In this paper, we are concerned with L^p–L^q estimate for T_λ with respect to general measures other than Lebesgue measure. More precisely, for $0<\alpha\le d$, let μ be a positive Borel regular measure which satisfies

$$\mu(B(x,\rho))\le C_\mu\rho^\alpha,\quad \rho>0 \tag{4}$$

for any $x\in\mathbb{R}^d$. Here the constant C_μ is independent of x, ρ. Considering $f=\chi_{[0,1]}$, one easily sees that the best possible β for (1) is α/q when $\nu\,(=\mu)$ satisfies (4). In fact, note that $|T_\lambda f(x)|\gtrsim 1$ if $|x|\le c\lambda^{-1}$ for a sufficiently small $c>0$. We aim to find the optimal range of (p,q) for which the inequality

$$\|T_\lambda^\gamma f\|_{L^q(d\mu)}\le C\lambda^{-\frac{\alpha}{q}}\|f\|_{L^p(I)} \tag{5}$$

holds under the assumption that μ satisfies (4).

In order to state our results we define a number $\beta=\beta(\alpha)$ by setting

$$\beta(\alpha)=(j+1)\alpha+\frac{(d-j-1)(d-j)}{2}$$

if $d-j-1<\alpha\le d-j$ for $j=0,\dots,d-1$. Note that $\beta(\alpha)$ increases by $j+1$ as α increases from $d-j-1$ to $d-j$. The following is our first result.

Theorem 1.1. *Let $\gamma\in C^{d+1}(I)$ and $0<\alpha\le d$ and let μ be a positive Borel regular measure. Suppose that γ and μ satisfies (2) and (4), respectively. Then for $1\le p,q\le\infty$ satisfying $d/q\le(1-1/p)$, $q\ge 2d$ and*

$$\beta(\alpha)/q+1/p<1,\ q>\beta(\alpha)+1,$$

there exists a constant C such that (5) holds for $f\in L^p(I)$ and $\lambda\ge 1$.

As α decreases the range of p, q gets larger (see Figure 1). If $\alpha = d$, this extends Drury's result [16] to general measures except for the end line case $\beta(d)/q + 1/p = 1$. (Note that $\beta(d) = (d^2+d)/2$.) The condition $d/q \le (1-1/p)$, $q \ge 2d$ comes from the use of Plancherel's theorem to obtain the d-linear operator (see Lemma 2.5). And the condition $\beta(\alpha)/q + 1/p < 1$ is sharp in that there is a measure satisfying (4) but (5) fails if $\beta(\alpha)/q + 1/p > 1$ (see Appendix A). The restriction $q > \beta(\alpha) + 1$ also seems sharp even though we presently don't have an example which shows it. Note that $\beta(\alpha) > d$ if $\alpha > 1$ and $\beta(\alpha)+1 > 2d$ if $\alpha > 2$. Hence, the assumption $d/q \le (1-1/p)$, $q \ge 2d$ is redundant when $\alpha > 2$. In particular, when μ is the surface measure on a compact smooth hypersurface $\Sigma \subset \mathbb{R}^d$ and $d \ge 3$, rescaling the estimate (5) we have the estimate

$$\|T_1 f\|_{L^q(\lambda\Sigma)} \le C\|f\|_{L^p(I)}$$

provided that $(d+2)(d-1)/(2q) + 1/p < 1$ and $p \le q$. This can be seen as a generalization of $L^p(S^1)$-$L^q(\lambda S^1)$ bound [9] (also see [21] and [4] for related results) for the extension operator from the circle S^1 in $\mathbb{R}^2$ to the large circle λS^1.

Our results here rely on so called *multilinear approach* which has been attempted to study restriction problem for hypersurfaces (*cf.* [2, 27]). Especially we adapt the recent argument due to Bourgain and Guth [10] (also see references therein) which was successful in deducing linear estimate from multilinear one. For the space curves sharp $d-$linear (extension) estimate is an easy consequence of Plancherel's theorem under the assumption that the support functions are separated (see Lemma 2.5 and Lemma 2.6). Then it is crucial to control $T_\lambda f$ by products of $T_\lambda f_i$ for which support of f_i is separated from each other while the remaining parts are involved with functions with relatively small supports (see Lemma 2.8). Compared with [10] this is relatively simpler since we only have to deal with one parameter separation in order to make use of multilinear estimate. To close the induction we need to obtain uniform estimates which do not depend on particular choice of curves. Hence after proper normalization of the curve we can reduce the matter to dealing with a class of curves which are close to a monomial curve. An obvious byproduct of this approach is the stability of estimates over a family of curves (see Remark 2.9).

The estimates of endpoint line case ($\beta(\alpha)/q + 1/p = 1$) are likely to be not possible with a general measure satisfying (4) except the trivial case. But they still look plausible with specific measures which satisfy certain regularity assumptions. However, these endpoint estimates are clearly beyond the reach of the method of this paper. On the other hand, one may try to use the method based on the offspring curves [16] but a routine adaptation of the presently known argument only gives (5) on restricted range, namely $d/q \le (1-1/p)$, $q \ge 2d$, $\beta(d)/q + 1/p < 1$ and $q > \beta(d) + 1$.

Finite type curves. There are also results when the curve degenerates, namely the condition (2) fails. Let us set $\mathbf{a} = (a_1, \cdots, a_d)$ with positive integers $a_1, a_2, \cdots, a_d$ satisfying $a_1 < a_2 < \cdots < a_d$. Then for $t \in I$ we also set

$$M_t^{\gamma,\mathbf{a}} = \left[\gamma^{(a_1)}(t), \quad \gamma^{(a_2)}(t), \quad \cdots, \quad \gamma^{(a_d)}(t)\right], \tag{6}$$

where the column vectors $\gamma^{(a_i)}(t)$ are a_i–th derivatives of γ. So, γ is nondegenerate at t if $\det M_t^{\gamma,\mathbf{a}} \neq 0$ with $\mathbf{a} = (1, 2, \cdots, d)$. We recall the following definition which was introduced in [11].

Definition 1.2. *Let $\gamma : I = [0, 1] \to \mathbb{R}^d$, $d \geq 2$ be a smooth curve. We say that γ is of finite type at $t \in I$ if there exists $\mathbf{a} = (a_1, \cdots, a_d)$ such that $\det M_t^{\gamma,\mathbf{a}} \neq 0$. We also say that γ is of finite type if so is γ at every $t \in I$.*

When degeneracy appears the boundedness of extension operator no longer remains the same so that (5) holds only on the smaller set of p, q. When μ is Lebesgue measure Christ [11] obtained sharp restriction estimates for curves of finite type on a restricted range. On the other hand, a natural attempt is to recover the full range (see (3)) by introducing a weight which mitigates bad behavior at degeneracy. In fact, let us consider the estimate

$$\|T_\lambda^\gamma[w, f]\|_{L^q(d\mu)} \leq C\lambda^{-\frac{\alpha}{q}} \|f\|_{L^p(wdt)},$$

where $\lambda \geq 1$ and

$$T_\lambda^\gamma[w, f](x) = \int_I e^{i\lambda x\cdot\gamma(t)} f(t)w(t)dt.$$

The dual form of this estimate with $\lambda = 1$ is

$$\Big(\int_I |\widehat{gd\mu}(\gamma(t))|^{p'} w(t)dt\Big)^{1/p'} \leq C\|g\|_{L^{q'}(d\mu)}. \tag{7}$$

There has been a long line of investigations on the estimate (7) [18, 19, 17, 5, 7, 6, 8, 15, 14] when the measure μ is Lebesgue measure and wdt is the affine arclength measure. When $d = 2$, the estimate was obtained by Sjölin [25] (also see [23]). In higher dimensions the study on (7) was initiated by Drury and Marshall [18], [19]. Drury [17], Bak and Oberlin [5] obtained partial results with some special curves in $\mathbb{R}^3$. If $I = \mathbb{R}$, by scaling the condition $d(d+1)/(2p') = 1/q$ is necessary for (7). Wright and Dendrinos [15] obtained a uniform estimate for a class of polynomial curves on the range $(d^2 + 2d)/2 < q \leq \infty$. This result was extended to a larger region [8] (see Section 8). There is also a result for the curves of which components are rational functions rather than polynomials (see [12]). Bak, Oberlin and Seeger obtained the estimates on the full range including the weak endpoint estimate for the monomial curves and the curves of simple type [8]. Dendrinos and Müller [14] further extended this result to the curves of small local perturbation of monomial curves and for the critical case $p = q = (d^2 + d + 2)/2$ the weak type endpoint estimate also holds for these curves (see *Remark* in Section 6 of [8]). The problem of obtaining (7) is now settled for finite type curves which are defined locally though uniform estimate is still open when curves are given on the whole real line.

In what follows we consider L^p–L^q estimate of $T_\lambda^\gamma[w_\gamma^\alpha, f]$ with respect to the measure μ satisfying (4). Let us define a measure by setting

$$w_\gamma^\alpha(t)dt = |\det(\gamma'(t), \gamma''(t), \cdots, \gamma^{(d)}(t))|^{\frac{1}{\beta(\alpha)}}dt.$$

When $\alpha = d$ this coincides with the affine arclength measure on γ. Considering a monomial curve and a measure satisfying the homogeneity condition $\int g(\lambda x)d\mu(x) =$

$\lambda^{-\alpha} \int g(x) d\mu(x)$ (for example the measure μ given in Appendix A), by rescaling one can easily see that the exponent $1/\beta(\alpha)$ is the natural choice in order that the estimate (8) holds for p, q satisfying $\beta(\alpha)/q+1/p \le 1$. In [8] (see Section 2), when μ is Lebesgue measure it was shown that the optimal power of torsion is $1/\beta(d) = 2/d(d+1)$ so that (7) holds for $d(d+1)/(2q) + 1/p \le 1$. If we consider the induced Lebesgue measure on lower dimensional planes, this shows that our choice of $\beta(\alpha)$ is optimal at least if α is an integer.

Our second result reads as follows.

Theorem 1.3. *Let $\gamma \in C^\infty(I)$ and $0 < \alpha \le d$. Suppose that μ satisfies (4) and γ is of finite type. Then, for $1 \le p, q \le \infty$ satisfying $d/q \le (1 - 1/p)$, $q \ge 2d$ and $\beta(\alpha)/q + 1/p < 1$, $q > \beta(\alpha) + 1$, there exists a constant C such that*

$$\|T^\gamma_\lambda [w^\alpha_\gamma, f]\|_{L^q(d\mu)} \le C\lambda^{-\frac{\alpha}{q}} \|f\|_{L^p(w^\alpha_\gamma dt)}. \tag{8}$$

This generalizes previous results to general measures except for p, q which are on the sharp line. Thanks to finite type assumption suitable normalization by a finite decomposition and rescaling reduce the problem to the case of monomial type curves of which degeneracy only appears a single point. Further decomposition away from degeneracy enables us to obtain the desired estimate (8) by relying on the stability of the estimate for non-degenerate curves.

The paper is organized as follows. In Section 2 we prove Theorem 1.1. In Section 3 we give the proof of Theorem 1.3 which is based on Theorem 1.1 and stability of the estimates. Sharpness of the condition $\beta(\alpha)/q+1/p < 1$ will be given in Appendix A.

2. Proof of Theorem 1.1

Proof of Theorem 1.1 is based on an adaptation of Bourgain-Guth argument in [10], which relies on a multilinear estimate and the uniform control of estimates over a class of curves and measures. This requires proper normalization of curves and measures.

Normalization of the curve. For $a, b \in \mathbb{R}$, $a \ne b$, we set

$$[a, b]^* = \begin{cases} [a, b] & \text{if } a < b, \\ [b, a] & \text{if } b < a. \end{cases}$$

Let $\gamma \in C^{d+1}(I)$ satisfying (2), and let $\tau \in I$ and h be a real number such that $[\tau, \tau + h]^* \subset I$. Then let us define $d \times d$ matrices M^γ_τ, D_h by

$$M^\gamma_\tau = M^{\gamma,(1,2,\dots,d)}_\tau = (\gamma'(\tau), \gamma''(\tau), \cdots, \gamma^{(d)}(\tau)),$$
$$D_h = (he_1, h^2 e_2, \dots, h^d e_d).$$

We also set

$$\gamma^h_\tau(t) = D_h^{-1} (M^\gamma_\tau)^{-1} (\gamma(ht + \tau) - \gamma(\tau)). \tag{9}$$

Then we have

$$x \cdot (\gamma(ht + \tau) - \gamma(\tau)) = D_h (M^\gamma_\tau)^t x \cdot \gamma^h_\tau(t). \tag{10}$$

Let us set

$$\gamma_\circ(t) = \Big(t, \frac{t^2}{2!}, \dots, \frac{t^d}{d!}\Big).$$

For a given $\epsilon > 0$ we define the class $\mathfrak{G}(\epsilon)$ of curves by setting

$$\Big\{\gamma \in C^{d+1}(I) : \|\gamma - \gamma_\circ\|_{C^{d+1}(I)} \le \epsilon\Big\}.$$

Lemma 2.1. *Let $\gamma \in C^{d+1}(I)$ satisfying (2) and let $\tau \in I$. Then, for $\epsilon > 0$ there is a $\delta > 0$ such that $\gamma_\tau^h \in \mathfrak{G}(\epsilon)$ whenever $[\tau, \tau + h]^* \subset I$ and $0 < |h| \le \delta$.*

For a given matrix M, $\|M\|$ denotes the usual matrix norm $\max_{|x|=1} |Mx|$.

Proof. It is enough to consider the case $[\tau, \tau + h] \subset I$. The other case $[h + \tau, \tau] \subset I$ can be shown similarly. By Taylor's expansion

$$\begin{aligned}\gamma(ht + \tau) - \gamma(\tau) &= \gamma'(\tau)ht + \gamma''(\tau)h^2\frac{t^2}{2!} + \dots + \gamma^{(d)}(\tau)h^d\frac{t^d}{d!} + \mathcal{E}(\tau, h, t)\\ &= M_\tau^\gamma D_h\gamma_\circ(t) + \mathcal{E}(\tau, h, t)\end{aligned}$$

with $\|\mathcal{E}(\tau, h, t)\|_{C^{d+1}(I)} \le Ch^{d+1}$ uniformly in τ. Since $\gamma_\tau^h(t) = \gamma_\circ(t) + (D_hM_\tau^\gamma)^{-1}$ $\mathcal{E}(\tau, h, t)$,

$$\|\gamma_\tau^h - \gamma_\circ\|_{C^{d+1}(I)} \le C\|(M_\tau^\gamma)^{-1}\|h. \tag{11}$$

By continuity $\|(M_\tau^\gamma)^{-1}\|$ obviously is uniformly bounded along $\tau \in I$ by a constant B because γ satisfies (2) and $\gamma \in C^{d+1}(I)$. Taking $\delta = \epsilon/(2CB)$, we see $\gamma_\tau^h \in \mathfrak{G}(\epsilon)$. $\square$

Remark 2.2. *Let $J \subset I$. From the proof of Lemma 2.1 it is clear that if $\|\gamma\|_{C^{d+1}(J)} \le B_1$ and $\|(M_\tau^\gamma)^{-1}\| \le B_2$ for all $\tau \in J$, then for any $\epsilon > 0$ there is a $\delta = \delta(B_1, B_2) > 0$ such that $\gamma_\tau^h \in \mathfrak{G}(\epsilon)$ provided that $[\tau, \tau + h]^* \subset J$ and $0 < |h| \le \delta$.*

Rescaling of measure. For $\mathbf{a} = (a_1, \cdots, a_d)$ let us define

$$D_h^{\mathbf{a}} = (h^{a_1}e_1, h^{a_2}e_2, \dots, h^{a_d}e_d). \tag{12}$$

For $M > 0$ we denote by $\mathfrak{B}(\alpha, M)$ the set of positive Borel regular measures which satisfy (4) with $C_\mu = M$. For a measure σ, $d \times d$ matrix A, and $0 < |h| < 1$ we define a measure $\sigma_{A,h}^{\mathbf{a}}$ by

$$\int F(x)d\sigma_{A,h}^{\mathbf{a}}(x) = \int F(AD_h^{\mathbf{a}}x)d\sigma(x). \tag{13}$$

Lemma 2.3. *Let $\mathbf{a} = (a_1, \cdots, a_d)$ and $a_1, a_2, \cdots, a_d$ satisfy that $0 < a_1 < a_2 < \cdots < a_d$ and $a_i \ge i$. If $\sigma \in \mathcal{B}(\alpha, M)$ and A is a nonsingular matrix, then $\sigma_{A,h}^{\mathbf{a}}$ is also a Borel regular measure satisfying*

$$\sigma_{A,h}^{\mathbf{a}}(B(x, \rho)) \le CM\|A^{-1}\|^\alpha|h|^{(\frac{d^2+d}{2} - \beta(\alpha) - \sum_{i=1}^d a_i)}\rho^\alpha \tag{14}$$

for $(x, \rho) \in \mathbb{R}^d \times \mathbb{R}_+$. Here C is independent of h, A.

Proof. Let $d-j-1<\alpha\le d-j$ for some $j=0,\dots,d-1$. By translation we may assume $x=0$. Let us set

$$S=\{y: AD_h^{\mathbf a}y\in B(0,\rho)\}.$$

We denote by $\omega_1,\cdots,\omega_d$ the row vectors of the nonsingular matrix A^{-1} and set $\|\omega\|=\max_k|\omega_k|$. Then, if $y=(y_1,\dots,y_d)\in S$ we have $|y_i|\le|\omega_i||h|^{-a_i}\rho\le\|\omega\||h|^{-a_i}\rho$. Hence, S is contained in the rectangle $\mathcal R$ of dimension $\|\omega\||h|^{-a_1}\rho\times\|\omega\||h|^{-a_2}\rho\times\cdots\times\|\omega\||h|^{-a_d}\rho$. So, $\mathcal R$ can be covered by as many as $O(|h|^{-a_1+1}\times|h|^{-a_2+2}\times\cdots\times|h|^{-a_d+d})$ rectangles $\mathcal R'$ of which dimension is $\|\omega\||h|^{-1}\rho\times\|\omega\||h|^{-2}\rho\times\cdots\times\|\omega\||h|^{-d}$ while each $\mathcal R'$ is covered by $O(1\times\cdots\times1\times|h|^{-1}\times\cdots\times|h|^{-(d-j-1)})$ cubes of sidelength $\|\omega\||h|^{-j-1}\rho$. Hence $\mathcal R$ is covered by cubes $\mathcal B_1,\dots,\mathcal B_l$ of sidelength $\|\omega\||h|^{-j-1}\rho$ with $l\lesssim|h|^{(\frac{d^2+d}{2}-\sum_{i=1}^d a_i-\frac{(d-j-1)(d-j)}{2})}$. So, it follows that

$$\begin{aligned}\sigma_{A,h}^{\mathbf a}(B(0,\rho))&\le\int\chi_{B(0,\rho)}(AD_h^{\mathbf a}y)d\sigma(y)\le\int\chi_{\mathcal R}(y)d\sigma(y)\\&\le\sum_{i=1}^l\int\chi_{\mathcal B_i}(y)d\sigma(y)=\sum_{i=1}^l\sigma(\mathcal B_i)\lesssim M\sum_{i=1}^l\|\omega\|^\alpha|h|^{-(j+1)\alpha}\rho^\alpha\\&\le CM\|A^{-1}\|^\alpha|h|^{(\frac{d^2+d}{2}-\sum_{i=1}^d a_i-\frac{(d-j-1)(d-j)}{2}-(j+1)\alpha)}.\end{aligned}$$

This gives the desired inequality since $(j+1)\alpha+(d-j-1)(d-j)/2=\beta(\alpha)$. $\square$

Multilinear ($d-$linear) estimates with separated supports. We now prove a multilinear estimate with respect to general measure, which is actually based on L^2 estimate. We also show that the estimates are uniform along $\gamma\in\mathfrak G(\epsilon)$ if $\epsilon>0$ is small enough. We start with proving the multilinear estimate which is basically a consequence of Plancherel's theorem.

Let us define a map $\Gamma_\gamma: I^d\to\mathbb R^d$ by

$$\Gamma_\gamma(\mathbf t)=\sum_{i=1}^d\gamma(t_i)$$

where $\mathbf t=(t_1,t_2,\cdots,t_d)$.

Lemma 2.4. *Let $E=\{\mathbf t\in I^d: t_1<t_2<\dots<t_d\}$. If $\epsilon>0$ is sufficiently small, then the map $\mathbf t: E\to\Gamma_\gamma(\mathbf t)$ is one to one for all $\gamma\in\mathfrak G(\epsilon)$.*

This can be shown by the argument in [19] which makes use of *total positivity* (also see [15]). In fact, we need to show that total positivity is valid on I regardless of $\gamma\in\mathfrak G(\epsilon)$. It is not difficult by making use of the fact that γ is a small perturbation of $\gamma_\circ$. We give a proof of Lemma 2.4 in Appendix B.

Now, the following is a straightforward consequence of Plancherel's theorem.

Lemma 2.5. *Let $\gamma\in\mathfrak G(\epsilon)$ and $\mathcal I_1,\dots,\mathcal I_d$ be closed intervals contained in I which satisfy* $\min_{i\ne j}\operatorname{dist}(\mathcal I_i,\mathcal I_j)\ge L$. *If $\epsilon>0$ is sufficiently small, then there is a constant*

C, independent of γ, such that

$$\|\prod_{i=1}^{d} T_\lambda^\gamma f_i\|_{L^2} \le CL^{-\frac{d^2-d}{4}}\lambda^{-\frac{d}{2}}\prod_{i=1}^{d}\|f_i\|_{L^2}$$

whenever f_i is supported in $\mathcal{I}_i$, $i=1,2,\dots,d$.

Proof. Let $f_1,\dots,f_d$ be supported in each interval $\mathcal{I}_i$, $i=1,2,\dots,d$, and set $F(\mathbf{t}) = \prod_{i=1}^d f_i(t_i)\chi_{\mathcal{I}_i}(t_i)$. Then, by Lemma 2.4 $\Gamma_\gamma : \prod_{i=1}^d \mathcal{I}_i \to \mathbb{R}^d$ is one to one. Hence by the change of variables $y = \Gamma_\gamma(\mathbf{t})$, we have

$$\prod_{i=1}^{d} T_\lambda^\gamma f_i = \int_{I^d} e^{i\lambda x\cdot\Gamma_\gamma(\mathbf{t})}F(\mathbf{t})d\mathbf{t} = \widehat{G}(\lambda x),$$

where $G(y) = F(\mathbf{t}(y))|\det\big(\frac{\partial\Gamma_\gamma}{\partial\mathbf{t}}\big)(\mathbf{t}(y))|^{-1}$. By reversing the change of variables $y\mapsto\mathbf{t}$ and Plancherel's theorem, we see

$$\|\prod_{i=1}^{d} T_\lambda^\gamma f_i\|_{L^2(\mathbb{R}^d)} = \lambda^{-\frac{d}{2}}\|G\|_{L^2(\mathbb{R}^d)} = \lambda^{-\frac{d}{2}}\bigg(\int |F(\mathbf{t})|^2\Big|\det\Big(\frac{\partial\Gamma_\gamma}{\partial\mathbf{t}}\Big)\Big|^{-1}d\mathbf{t}\bigg)^{\frac12}.$$

Since $|t_j - t_i| \ge L$, $i\ne j$, it is sufficient to show that for all $\gamma\in\mathfrak{G}(\epsilon)$

$$\Big|\det\Big(\frac{\partial\Gamma_\gamma}{\partial\mathbf{t}}\Big)\Big| \ge \frac{1}{2\prod_{i=1}^d (i-1)!}\prod_{1\le i<j\le d}|t_j-t_i|$$

if ϵ is sufficiently small. If $\gamma\in\mathfrak{G}(\epsilon)$, then $\gamma = \gamma_\circ + \mathcal{E}$ and $\|\mathcal{E}\|_{C^{d+1}(I)}\le\epsilon$. Hence by a computation with a generalized mean value theorem we see that

$$\Big|\det\Big(\frac{\partial\Gamma_\gamma}{\partial\mathbf{t}}\Big)\Big| \ge \frac{1}{\prod_{i=1}^d (i-1)!}\prod_{1\le i<j\le d}|t_j-t_i| \times (1-\epsilon 2^{d-1}d!).$$

Taking a small ϵ so that $\epsilon < (2^d d!)^{-1}$, we get the desired estimate. This completes the proof. □

Using Lemma 2.5, we obtain the following L^p–L^q estimate via interpolation with L^1–L^∞ estimate.

Proposition 2.6. *Let $\mathcal{I}_1,\dots,\mathcal{I}_d$, and $\gamma\in\mathfrak{G}(\epsilon)$ be given as in Lemma 2.5. Suppose μ satisfies* (4)*. If $\epsilon>0$ is sufficiently small, then for $1/p+1/q\le 1$ and $q\ge 2$ there is a constant C, independent of γ, such that*

$$\|\prod_{i=1}^{d} T_\lambda^\gamma f_i\|_{L^q(d\mu)} \le CC_\mu^{\frac1q}L^{-\frac{d^2-d}{2q}}\lambda^{-\frac{\alpha}{q}}\prod_{i=1}^{d}\|f_i\|_p$$

whenever f_i is supported in $\mathcal{I}_i$, $i=1,2,\dots,d$.

Proof. To begin with, we observe that the trivial L^1–L^∞ estimate

$$\|\prod_{i=1}^{d} T_\lambda^\gamma f_i\|_{L^\infty(d\mu)} \le \prod_{i=1}^{d}\|f_i\|_1$$

holds. It is obvious because $\prod_{i=1}^d T_\lambda^\gamma f_i$ is continuous and $|\prod_{i=1}^d T_\lambda^\gamma f_i| \le \prod_{i=1}^d \|f_i\|_1$. Since f_i is supported in $\mathcal{I}_i \subset I$, by Hölder's inequality and interpolation, it suffices to show that

$$\|\prod_{i=1}^d T_\lambda^\gamma f_i\|_{L^2(d\mu)} \le C C_\mu^{\frac12} L^{-\frac{d^2-d}{4}} \lambda^{-\frac{\alpha}{2}} \prod_{i=1}^d \|f_i\|_2. \tag{15}$$

Observe that the Fourier transform $\mathcal{F}(\prod_{i=1}^d T_\lambda^\gamma f_i)$ of $\prod_{i=1}^d T_\lambda^\gamma f_i$ is supported in a ball of radius $C\sqrt{2d}\,\lambda$ for some C. Let φ be a smooth function such that $\widehat{\varphi} = 0$ if $|\xi| \ge 2C\sqrt{2d}$ and $\widehat{\varphi} = 1$ if $|\xi| \le C\sqrt{2d}$. Consequently, $\mathcal{F}(\prod_{i=1}^d T_\lambda^\gamma f_i)(\xi) = \mathcal{F}(\prod_{i=1}^d T_\lambda^\gamma f_i)(\xi)\widehat{\varphi}(\xi/\lambda)$. Hence

$$\prod_{i=1}^d T_\lambda^\gamma f_i = \varphi_\lambda * (\prod_{i=1}^d T_\lambda^\gamma f_i)$$

where $\varphi_\lambda(x) = \lambda^d \varphi(\lambda x)$. In addition, $|\prod_{i=1}^d T_\lambda^\gamma f_i|^2 \le C|\prod_{i=1}^d T_\lambda^\gamma f_i|^2 * |\varphi|_\lambda$ with C only depending on φ. Using the rapid decay of φ and (4), we see

$$|\varphi_\lambda| * \mu = \int \lambda^d |\varphi|(\lambda(x-y))d\mu(y) \le C C_\mu \lambda^{d-\alpha}.$$

Therefore, by using this and Fubini's theorem

$$\begin{aligned}\|\prod_{i=1}^d T_\lambda^\gamma f_i\|_{L^2(d\mu)} &\le \Big(\int |\prod_{i=1}^d T_\lambda^\gamma f_i|^2 * |\varphi|_\lambda d\mu(x)\Big)^{\frac12} \le \|\prod_{i=1}^d T_\lambda f_i\|_2 \, \||\varphi|_\lambda * \mu\|_\infty^{\frac12} \\ &\lesssim C_\mu^{\frac12} L^{-\frac{d^2-d}{4}} \lambda^{-\frac{d}{2}} \lambda^{\frac{d}{2}-\frac{\alpha}{2}} \prod_{i=1}^d \|f_i\|_2 \lesssim C_\mu^{\frac12} L^{-\frac{d^2-d}{4}} \lambda^{-\frac{\alpha}{2}} \prod_{i=1}^d \|f_i\|_2.\end{aligned}$$

For the third inequality we use Lemma 2.5. Hence we get (15). □

The induction quantity. For $1 \le \lambda$, $1 \le p, q \le \infty$, and $\epsilon > 0$, we define $Q_\lambda(R) = Q_\lambda(R, p, q, \epsilon)$ by setting

$$Q_\lambda(R) = \sup\{\ \|T_\lambda^\gamma f\|_{L^q(d\mu, B_R)} : \mu \in \mathfrak{B}(\alpha, 1),\ \gamma \in \mathfrak{G}(\epsilon),\ \|f\|_{L^p(I)} \le 1\} \tag{16}$$

where B_R is the ball of radius R centred at the origin. Clearly, $Q_\lambda(R) < \infty$ because $Q_\lambda(R) \le R^{\alpha/q}$ for any $\lambda > 0$.

Lemma 2.7. *Let $\gamma \in \mathfrak{G}(\epsilon)$, $\mu \in \mathfrak{B}(\alpha, 1)$, and let $\lambda \ge 1$, $0 < |h| < 1$. Suppose that f is supported in the interval $[\tau, \tau + h]^* \subset [0, 1]$. Then, if $\epsilon > 0$ is sufficiently small, there is a constant $\delta > 0$, independent of γ, such that if $0 < |h| \le \delta$*

$$\|T_\lambda^\gamma f\|_{L^q(d\mu, B_R)} \le C\,|h|^{1-\frac1p-\frac{\beta(\alpha)}{q}} Q_\lambda(R)\|f\|_p. \tag{17}$$

Proof. We begin with setting $f_h = |h| f(ht + \tau)$. By translation, scaling and using (10) it follows that

$$|T_\lambda^\gamma f(x)| = \Big|\int_I e^{i\lambda x\cdot(\gamma(ht+\tau)-\gamma(\tau))} f_h(t)dt\Big| = \Big|\int_I e^{i\lambda D_h (M_\tau^\gamma)^t x\cdot \gamma_\tau^h(t)} f_h(t)dt\Big|. \tag{18}$$

We denote by μ_τ^h the measure given by

$$\int F(x)d\mu_\tau^h(x) = \int F(D_h(M_\tau^\gamma)^t x)d\mu(x) \tag{19}$$

and set

$$d\widetilde{\mu}(x) = \frac{|h|^{\beta(\alpha)}}{1 + C\|(M_\tau^\gamma)^{-t}\|^\alpha}\, d\mu_\tau^h(x).$$

Hence, we have

$$\begin{aligned}\int_{B_R} |T_\lambda^\gamma f(x)|^q d\mu(x) = \int |T_\lambda^{\gamma_\tau^h} f_h(x)|^q \chi_{B_R}((M_\tau^\gamma)^{-t} D_h^{-1} x) d\mu_\tau^h(x)\\ =(1 + C\|(M_\tau^\gamma)^{-t}\|^\alpha)|h|^{-\beta(\alpha)} \int |T_\lambda^{\gamma_\tau^h} f_h(x)|^q \chi_{B_R}((M_\tau^\gamma)^{-t} D_h^{-1} x) d\widetilde{\mu}(x)\end{aligned}$$

Now we note that $\|(M_\tau^\gamma)^{-t}\| \le C$ uniformly for $\gamma \in \mathfrak{G}(\epsilon)$ if $\epsilon > 0$ is small enough. By Lemma 2.3 $\widetilde{\mu} \in \mathfrak{B}(\alpha, 1)$, and $\gamma_\tau^h \in \mathfrak{G}(C|h|\epsilon) \subset \mathfrak{G}(\epsilon)$ if $0 < |h| \le \delta$ for a sufficiently small $\delta > 0$. Moreover, the set $\{M_\tau^\gamma D_h x : x \in B_R\}$ is also contained in B_R for all $\gamma \in \mathfrak{G}(\epsilon)$ if $0 < |h| \le \delta$ and δ is small enough. Therefore, by the definition of $Q_\lambda(R)$ we see

$$\begin{aligned}\int_{B_R} |T_\lambda^\gamma f(x)|^q d\mu(x) &\le C|h|^{-\beta(\alpha)} \int_{B_R} |T_\lambda^{\gamma_\tau^h} f_h(x)|^q d\widetilde{\mu}(x)\\ &\le C|h|^{-\beta(\alpha)} (Q_\lambda(R)\, \|f_h\|_p)^q\\ &= C|h|^{-\beta(\alpha)+q-\frac{q}{p}} (Q_\lambda(R)\|f\|_p)^q.\end{aligned}$$

This gives the desired inequality (17). □

Multilinear decomposition. Now we make decomposition of T_λ^γ which is needed to exploit the d-linear estimate. This decomposition doesn't depend on particular choice of γ.

Let $A_1, \dots, A_{d-1}$ be dyadic numbers such that

$$1 = A_0 \gg A_1 \gg A_2 \cdots \gg A_{d-1}.$$

For $i = 1, \dots, d-1$, let us denote by $\{\mathfrak{I}^i\}$ the collection of dyadic intervals of length A_i which are contained in $[0, 1]$. And we set

$$f_{\mathfrak{I}^i} = \chi_{\mathfrak{I}^i} f$$

so that for $i = 1, \dots, d-1$,

$$f = \sum_{\mathfrak{I}^i} f_{\mathfrak{I}^i}$$

whenever f is supported in I. Let $S_1, \dots, S_i$ be subsets of I and let us define

$$\Delta(S_1, S_2, \dots, S_i) = \min_{j \ne k} \operatorname{dist}(S_j, S_k).$$

Lemma 2.8. *Let $\gamma : I \to \mathbb{R}^d$ be a smooth curve. Let $A_0, A_1, \dots, A_{d-1}$, and $\{\mathfrak{I}^i\}$, $i = 1, \dots, d-1$ be defined as in the above. Then, for any $x \in \mathbb{R}^d$, there is a constant C, independent of γ, x, $A_0, A_1, \dots, A_{d-1}$, such that*

$$\begin{aligned}|T_\lambda^\gamma f(x)| \le \sum_{i=1}^{d-1} C A_{i-1}^{-2(i-1)} \max_{\mathfrak{I}^i} |T_\lambda^\gamma f_{\mathfrak{I}^i}(x)| \\ + C A_{d-1}^{-2(d-1)} \max_{\Delta(\mathfrak{I}_1^{d-1}, \mathfrak{I}_2^{d-1}, \dots, \mathfrak{I}_d^{d-1}) \ge A_{d-1}} |\prod_{i=1}^{d} T_\lambda^\gamma f_{\mathfrak{I}_i^{d-1}}(x)|^{\frac{1}{d}}.\end{aligned} \tag{20}$$

The exact exponents of A_i are not important for the argument here. So, we don't try to obtain most efficient exponents.

Proof. Fix $x \in \mathbb{R}^d$. By a simple argument it is easy to see that

$$|T_\lambda^\gamma f(x)| \le C \max_{\mathfrak{I}^1} |T_\lambda^\gamma f_{\mathfrak{I}^1}(x)| + C A_1^{-1} \max_{\Delta(\mathfrak{I}_1^1, \mathfrak{I}_2^1) \ge A_1} |T_\lambda^\gamma f_{\mathfrak{I}_1^1}(x) T_\lambda^\gamma f_{\mathfrak{I}_2^1}(x)|^{\frac{1}{2}}.$$

In fact, let $\mathfrak{I}_*^1$ be the interval such that $|T_\lambda^\gamma f_{\mathfrak{I}_*^1}(x)| = \max_{\mathfrak{I}^1} |T_\lambda^\gamma f_{\mathfrak{I}^1}(x)|$. Then we consider the cases $|T_\lambda^\gamma f(x)| \le 100 |T_\lambda^\gamma f_{\mathfrak{I}_*^1}(x)|$, $|T_\lambda^\gamma f(x)| \ge 100 |T_\lambda^\gamma f_{\mathfrak{I}_*^1}(x)|$, separately.

For the latter case, there is a $\mathfrak{I}^1$ such that $|T_\lambda^\gamma f(x)| \le C A_1^{-1} |T_\lambda^\gamma f_{\mathfrak{I}^1}(x)|$ and $\Delta(\mathfrak{I}_*^1, \mathfrak{I}^1) \ge A_1$. Then it follows that

$$|T_\lambda^\gamma f(x)| \le C A_1^{-1} |T_\lambda^\gamma f_{\mathfrak{I}^1}(x) T_\lambda^\gamma f_{\mathfrak{I}_*^1}(x)|^{\frac{1}{2}}.$$

Combining the above two cases we get the desired inequality.

Now, for $j \ge 2$ we claim that

$$\begin{aligned}&\max_{\Delta(\mathfrak{I}_1^{j-1}, \mathfrak{I}_2^{j-1}, \dots, \mathfrak{I}_j^{j-1}) \ge A_{j-1}} |\prod_{i=1}^{j} T_\lambda^\gamma f_{\mathfrak{I}_i^{j-1}}(x)|^{\frac{1}{j}} \\ \le& C \max_{\mathfrak{I}^j} |T_\lambda f_{\mathfrak{I}^j}(x)| + C A_j^{-2} \max_{\Delta(\mathfrak{I}_1^j, \mathfrak{I}_2^j, \dots, \mathfrak{I}_{j+1}^j) \ge A_j} |\prod_{i=1}^{j+1} T_\lambda^\gamma f_{\mathfrak{I}_i^j}(x)|^{\frac{1}{j+1}}\end{aligned} \tag{21}$$

holds with C, independent of A_{j-1}, A_j. This actually proves the desired inequality. Applying (21) successively we see that

$$\begin{aligned}|T_\lambda^\gamma f(x)| \le C \sum_{i=1}^{d-1} \Big[\prod_{k=1}^{i} A_{k-1}^{-2}\Big] \max_{\mathfrak{I}^i} |T_\lambda^\gamma f_{\mathfrak{I}^i}(x)| \\ + C \Big[\prod_{k=1}^{d-1} A_{k-1}^{-2}\Big] \max_{\Delta(\mathfrak{I}_1^{d-1}, \mathfrak{I}_2^{d-1}, \dots, \mathfrak{I}_d^{d-1}) \ge A_{d-1}} |\prod_{i=1}^{d} T_\lambda^\gamma f_{\mathfrak{I}_i^{d-1}}(x)|^{\frac{1}{d}}.\end{aligned}$$

Obviously this implies (20). Hence it remains to show (21).

Suppose that intervals $\mathfrak{I}_1^{j-1}, \mathfrak{I}_2^{j-1}, \dots, \mathfrak{I}_j^{j-1}$ of length A_{j-1} with $\Delta(\mathfrak{I}_1^{j-1}, \mathfrak{I}_2^{j-1}, \dots, \mathfrak{I}_j^{j-1}) \ge A_{j-1}$ are given. Then let us denote by $\{\mathfrak{I}_i^j\}$, $i = 1, \dots, j$, the dyadic interval of sidelength A_j such that $\mathfrak{I}_i^j \subset \mathfrak{I}_i^{j-1}$. Also let us denote by $\mathfrak{I}_{i,*}^j$ the interval such that

$$|T_\lambda^\gamma f_{\mathfrak{I}_{i,*}^j}(x)| = \max_{\mathfrak{I}_i^j} |T_\lambda^\gamma f_{\mathfrak{I}_i^j}(x)|.$$

Then we consider the following two cases:

$$\text{(I)} \qquad \begin{aligned} &|T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)| < A_j^j \max_{i=1,\dots,j} \max_{\mathfrak{I}^j_i} |T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)| \text{ for some } i, \\ \text{or} \quad & \\ &\Delta(\mathfrak{I}^j_i, \mathfrak{I}^j_{i,*}) < A_j \text{ for all } i, \end{aligned}$$

and

$$\text{(II)} \qquad \begin{aligned} &|T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)| \ge A_j^j \max_{i=1,\dots,j} \max_{\mathfrak{I}^j_i} |T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)| \text{ for all } i, \\ \text{and} \quad & \\ &\Delta(\mathfrak{I}^j_i, \mathfrak{I}^j_{i,*}) \ge A_j \text{ for some } i. \end{aligned}$$

We now split

$$\prod_{i=1}^{j} T^\gamma_\lambda f_{\mathfrak{I}^{j-1}_i}(x) = \sum_{\mathfrak{I}^j_1,\dots,\mathfrak{I}^j_j} \prod_{i=1}^{j} T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x) = \Big(\sum_{\text{(I)}} + \sum_{\text{(II)}}\Big) \prod_{i=1}^{j} T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x).$$

Since $\#\{\mathfrak{I}^j_i\} \le A_j^{-1}$, there are $O(A_j^{-j})$ j-tuples $(\mathfrak{I}^j_1, \dots, \mathfrak{I}^j_j)$ in the summation of the case (I). Hence it is easy to see that

$$\Big|\sum_{\text{(I)}} \prod_{i=1}^{j} T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)\Big|^{\frac{1}{j}} \le C \max_{i=1,\dots,j} \max_{\mathfrak{I}^j_i} |T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)|. \tag{22}$$

Now let us consider a term $\prod_{i=1}^j T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)$ from the second case (II). Then there is an $\mathfrak{I}^j_i$ such that $\Delta(\mathfrak{I}^j_i, \mathfrak{I}^j_{i,*}) \ge A_j$. Since $|T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)| \ge A_j^j \max_{i=1,\dots,j} \max_{\mathfrak{I}^j_i} |T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)|$ for all i, using the fact that $\Delta(\mathfrak{I}^j_{1,*}, \dots, \mathfrak{I}^j_{j,*}) \ge A_{j-1}$ we see that

$$\begin{aligned} \prod_{k=1}^{j} |T^\gamma_\lambda f_{\mathfrak{I}^j_k}(x)|^{\frac{1}{j}} &\le A_j^{\frac{-j}{j+1}} |T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)|^{\frac{1}{j+1}} \prod_{k=1}^{j} |T^\gamma_\lambda f_{\mathfrak{I}^j_{k,*}}(x)|^{\frac{1}{j+1}} \\ &\le A_j^{-1} \max_{\Delta(\mathfrak{I}^j_1, \mathfrak{I}^j_2, \dots, \mathfrak{I}^j_{j+1}) \ge A_j} |\prod_{i=1}^{j+1} T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)|^{\frac{j}{j+1}}. \end{aligned}$$

Since there are $O(A^{-j})$ j-tuples $(\mathfrak{I}^j_1, \dots, \mathfrak{I}^j_j)$, it follows that

$$\Big|\sum_{\text{(II)}} \prod_{i=1}^{j} T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)\Big|^{\frac{1}{j}} \le C A_j^{-2} \max_{\Delta(\mathfrak{I}^j_1, \mathfrak{I}^j_2, \dots, \mathfrak{I}^j_{j+1}) \ge A_j} |\prod_{i=1}^{j+1} T^\gamma_\lambda f_{\mathfrak{I}^j_i}(x)|^{\frac{j}{j+1}}.$$

Combining this with (22) gives (21). □

Proof of Theorem 1.1. Since $\gamma \in C^{d+1}(I)$ satisfies (2), by continuity it follows that there is a constant C_γ such that

$$\|(M^\gamma_\tau)^{-1}\| \le C_\gamma, \ \tau \in I.$$

Let $0 < \epsilon \le 1$ be a small number so that Proposition 2.6 and Lemma 2.7 holds. Then fix $0 < \delta < 1$ such that Lemma 2.1 holds.

Fixing an integer ℓ satisfying $1/\ell < \delta$, we now break the interval I such that $I = \cup_{j=0}^{\ell-1}[\frac{j}{\ell}, \frac{j+1}{\ell}]$. Then let us set $h = 1/\ell$ and

$$f_j(t) = hf(ht + jh)\chi_I.$$

Recalling (9) and (19), for $j = 0, \dots, \ell - 1$ we also set

$$\gamma_j = \gamma_{jh}^h, \quad \mu_j = \frac{1}{C_{\gamma,j,h}}\, \mu_{jh}^h,$$

where μ_{jh}^h is defined by (19) and $C_{\gamma,j,h} = (1 + C\|(M_\tau^\gamma)^{-t}\|^\alpha)h^{-\beta(\alpha)}$.

Now by Lemma 2.1 it follows that $\gamma_j \in \mathfrak{G}(\epsilon)$ and by Lemma 2.3 we see that $\mu_j \in \mathfrak{B}(\alpha, 1)$. Hence, after rescaling (see (2.3)) we have

$$\|T_\lambda^\gamma f\|_{L^q(d\mu)} \le \sum_{j=0}^{\ell-1} \|T_\lambda^\gamma f\chi_{[jh,(j+1)h]}\|_{L^q(d\mu)} = (C_{\gamma,j,h})^{\frac{1}{q}} \sum_{j=0}^{\ell-1} \|T_\lambda^{\gamma_j} f_j\|_{L^q(d\mu_j)}. \tag{23}$$

Therefore for the proof of Theorem 1.1 it is sufficient to show (5) when $\gamma \in \mathfrak{G}(\epsilon)$, $\mu \in \mathfrak{B}(\alpha, 1)$.

Let $p \ge 1$, $q \ge 1$ satisfy $d/q \le (1-1/p)$, $q \ge 2d$, and $\beta(\alpha)/q+1/p < 1$, $q > \beta(\alpha)+1$. Since I is compact, it is enough to consider $q \ge p$. Let $\gamma \in \mathfrak{G}(\epsilon)$, $\mu \in \mathfrak{B}(\alpha, 1)$, and f be a function supported in I with $\|f\|_{L^p(I)} = 1$ such that

$$Q_\lambda(R) = Q_\lambda(R, p, q) \le 2\|T_\lambda^\gamma f\|_{L^q(d\mu, B_R)}.$$

Set $A_0 = 1$ and let $A_1, \dots, A_{d-1}$ be dyadic numbers such that $\delta \gg A_1 \gg A_2 \cdots \gg A_{d-1}$. These numbers will be chosen later. Then, recalling (20), using Lemma 2.7, and noting $q \ge p$, we see that

$$\begin{aligned}
&\Big\|\max_{\mathfrak{J}^i} |T_\lambda^\gamma f_{\mathfrak{J}^i}|\Big\|_{L^q(d\mu,B_R)} \le \Big(\sum_{\mathfrak{J}^i} \Big\|T_\lambda^\gamma f_{\mathfrak{J}^i}\Big\|_{L^q(d\mu,B_R)}^q\Big)^{\frac{1}{q}} \\
&\le {A_i}^{1-\frac{1}{p}-\frac{\beta(\alpha)}{q}} Q_\lambda(R)\Big(\sum_{\mathfrak{J}^i} \|f_{\mathfrak{J}^i}\|_p^q\Big)^{\frac{1}{q}} \le {A_i}^{1-\frac{1}{p}-\frac{\beta(\alpha)}{q}} Q_\lambda(R)\Big(\sum_{\mathfrak{J}^i} \|f_{\mathfrak{J}^i}\|_p^p\Big)^{\frac{1}{p}} \\
&= {A_i}^{1-\frac{1}{p}-\frac{\beta(\alpha)}{q}} Q_\lambda(R)\|f\|_p\,.
\end{aligned}$$

And using Proposition 2.6, we also have

$$\|\max_{\Delta(\mathfrak{J}_1^{d-1},\mathfrak{J}_2^{d-1},\dots,\mathfrak{J}_d^{d-1})\ge A_{d-1}} |\prod_{i=1}^d T_\lambda^\gamma f_{\mathfrak{J}_i^{d-1}}|^{\frac{1}{d}}\|_{L^q(d\mu)} \le CA_{d-1}^{-C}\lambda^{-\frac{\alpha}{q}}\|f\|_p.$$

Using the decomposition (20) and combining the above two estimates, we see that

$$\|T_\lambda^\gamma f\|_{L^q(d\mu)} \le C\sum_{i=1}^{d-1} A_{i-1}^{-C}{A_i}^{1-\frac{1}{p}-\frac{\beta(\alpha)}{q}} Q_\lambda(R)\|f\|_p + CA_{d-1}^{-C}\lambda^{-\frac{\alpha}{q}}\|f\|_p$$

holds independent of $\gamma \in \mathfrak{G}(\epsilon)$, $\mu \in \mathfrak{B}(\alpha, 1)$. Hence, taking sup with respect to f with $\|f\|_p \le 1$, $\gamma \in \mathfrak{G}(\epsilon)$, and $\mu \in \mathfrak{B}(\alpha, 1)$, we see that

$$Q_\lambda(R) \le C\sum_{i=1}^{d-1} A_{i-1}^{-C}{A_i}^{1-\frac{1}{p}-\frac{\beta(\alpha)}{q}} Q_\lambda(R) + CA_{d-1}^{-C}\lambda^{-\frac{\alpha}{q}}$$

by the definition of $Q_\lambda(R)$. Hence this gives

$$\lambda^{\frac{\alpha}{q}} Q_\lambda(R) \le C \sum_{i=1}^{d-1} A_{i-1}^{-C} A_i^{1-\frac{1}{p}-\frac{\beta(\alpha)}{q}} \lambda^{\frac{\alpha}{q}} Q_\lambda(R) + C A_{d-1}^{-C}.$$

Since $1-\frac{1}{p}-\frac{\beta(\alpha)}{q} > 0$, we can successively choose $A_1, \dots, A_{d-1}$ so that $C A_{i-1}^{-C} A_i^{1-\frac{1}{p}-\frac{\beta(\alpha)}{q}} < \frac{1}{2d}$ for $i = 1, \dots, d-1$. Hence we get

$$\lambda^{\frac{\alpha}{q}} Q_\lambda(R) \le \frac{1}{2} \lambda^{\frac{\alpha}{q}} Q_\lambda(R) + C A_{d-1}^{-C}$$

whenever $1 \le \lambda$. Hence it follows that $\lambda^{\frac{\alpha}{q}} Q_\lambda(R) \le C A_{d-1}^{-C}$ provide that $1 - \frac{1}{p} - \frac{\beta(\alpha)}{q} > 0$. Hence $Q_\lambda(R) \le C\lambda^{-\frac{\alpha}{q}}$. Letting $R \to \infty$ completes the proof. $\square$

Remark 2.9. *Note that the estimates in Proposition 2.6 and Lemma 2.7 hold uniformly for all $\gamma \in \mathfrak{G}(\epsilon)$, $\mu \in \mathfrak{B}(\alpha, 1)$ if $\epsilon > 0$ is sufficiently small, and Lemma 2.8 remains valid regardless of particular $\gamma \in \mathfrak{G}(\epsilon)$. Hence the last part of the proof of Theorem 1.1 actually shows that there is a constant C, independent of γ, μ, such that*

$$\|T_\lambda^\gamma f\|_{L^q(d\mu)} \le C\lambda^{-\frac{\alpha}{q}} \|f\|_{L^p(I)}$$

provided that $\gamma \in \mathfrak{G}(\epsilon)$, $\mu \in \mathfrak{B}(\alpha, 1)$ and $\epsilon > 0$ is sufficiently small.

3. Proof of Theorem 1.3; Finite type curves

As in the non-degenerate case, if the curve satisfies finite condition it can locally be considered as a perturbation of monomial curves. The following is a simple consequence of Taylor's theorem.

Lemma 3.1. *Let $\gamma : I = [0,1] \to \mathbb{R}^d$ be a smooth curve. Suppose that γ is of finite type at $\tau \in I$. Then there exist $\delta > 0$ and $\mathbf{a} = (a_1, \cdots, a_d)$ with positive integers $a_1, a_2, \cdots, a_d$ satisfying $a_1 < a_2 < \cdots < a_d$, such that $M_\tau^{\gamma,\mathbf{a}}$ is nonsingular and*

$$\gamma(t+\tau) - \gamma(\tau) = M_\tau^{\gamma,\mathbf{a}}(t^{a_1}\varphi_1(t), t^{a_2}\varphi_2(t), \cdots, t^{a_d}\varphi_d(t)), \tag{24}$$

for $t \in [-\delta, \delta] \cap (I - \tau)$ where φ_k is a smooth function satisfying

$$(t^{a_k}\varphi_k)^{(a_j)}(0) = \delta_{j,k} \text{ for } 1 \le j, k \le d\,. \tag{25}$$

The last condition (25) implies that $\varphi_k(0) = 1/(a_k!)$ for $1 \le k \le d$. Furthermore it is easy to see that $\mathbf{a}$ and $\varphi_1(t), \dots, \varphi_d(t)$ are uniquely determined. In fact, suppose that

$$M(t^{a_1}\varphi_1(t), \cdots, t^{a_d}\varphi_d(t)) = M'(t^{b_1}\widetilde{\varphi}_1(t), \cdots, t^{b_d}\widetilde{\varphi}_d(t))$$

for nonsingular matrices M, M', positive integers $b_1 < b_2 <, \dots, < b_d$ and smooth $\widetilde{\varphi}_i$ with $(t^{b_k}\widetilde{\varphi}_k)^{(b_j)}(0) = \delta_{j,k}$ for $1 \le j, k \le d$. Now let M_k and M_k' denote the $k-$th column of matrices M and M', respectively. The above is now written as

$$M_1 t^{a_1}\varphi_1(t) + \cdots + M_d t^{a_d}\varphi_d(t) = M_1' t^{b_1}\widetilde{\varphi}_1(t) + \cdots + M_d' t^{b_d}\widetilde{\varphi}_d(t).$$

Differentiating a_1 times at $t = 0$, we see $b_1 \le a_1$. By symmetry we also have $b_1 \ge a_1$. Hence $a_1 = b_1$ and by (25) we see that $M_1 = M_1'$. Similarly, by differentiating a_2 times at $t = 0$ and using (25) it follows that $a_2 = b_2$ and $M_2 = M_2'$. By repeating

this we see that $a_1 = b_1, \dots, a_d = b_d$ and $M_1 = M_1', \dots, M_d = M_d'$. Then, since $M_1, \dots, M_d$ are linearly independent, it follows that $\varphi_i(t) = \widetilde{\varphi}_i(t)$ for $i = 1, \dots, d$.

Definition 3.2. *Let $\gamma : I = [0,1] \to \mathbb{R}^d$ be a smooth curve and $\tau \in I$. If there are a nonsingular matrix M, positive integers $a_1, a_2, \cdots, a_d$ with $a_1 < a_2 < \cdots < a_d$ and smooth functions $\varphi_1, \dots, \varphi_d$ satisfying* (25) *such that*

$$\gamma(t+\tau) - \gamma(\tau) = M(t^{a_1}\varphi_1(t), t^{a_2}\varphi_2(t), \cdots, t^{a_d}\varphi_d(t)),\ t \in [-\delta, \delta] \cap (I - \tau)$$

for some $\delta > 0$, then we say that γ is of type $\mathbf{a}$ at $t = \tau$.

Proof of Lemma (3.1). Let a_1 be the smallest integer such that $\gamma^{(a_1)}(\tau) \neq 0$. Then let a_2 be the smallest integer such that $\gamma^{(a_1)}(\tau)$ and $\gamma^{(a_2)}(\tau)$ are linearly independent. Now we inductively choose a_j to be the smallest integer such that $\gamma^{(a_1)}(\tau), \dots, \gamma^{(a_{j-1})}(\tau), \gamma^{(a_j)}(\tau)$ are linearly independent. Since γ is finite type at τ, this gives linearly independent vectors $\gamma^{(a_1)}(\tau), \dots, \gamma^{(a_d)}(\tau)$.

Let us set $a_0 = 0$. Then for $j = 1, \dots, d$, it follows that if $a_{j-1} < \ell < a_j$

$$\gamma^{(\ell)}(\tau) \in \operatorname{span}\{\gamma^{(a_1)}(\tau), \dots, \gamma^{(a_{j-1})}(\tau)\}. \tag{26}$$

Now by Taylor expansion of $\gamma(t)$ at $t = \tau$, we write

$$\gamma(t+\tau) = \gamma(\tau) + \sum_{\ell=1}^{a_d} \frac{t^\ell}{\ell!}\gamma^{(\ell)}(\tau) + \frac{t^{a_d+1}}{(a_d+1)!}\,\mathcal{E}(\mathbf{c})$$

where $\mathcal{E}(\mathbf{c}) = (\gamma_1^{(a_d+1)}(c_1), \dots, \gamma_d^{(a_d+1)}(c_d))$ and $\mathbf{c} = (c_1, \dots, c_d)$ with $c_i \in (\tau, t+\tau)$, $1 \le i \le d$. Then by (26) it follows that for $j = 2, \dots, d$

$$\sum_{\ell=a_{j-1}}^{a_j-1} \frac{t^\ell}{\ell!}\gamma^{(\ell)}(\tau) = \frac{t^{a_{j-1}}}{a_{j-1}!}\gamma^{(a_{j-1})}(\tau) + \sum_{k=1}^{j-1} p_{j,k}(t)\gamma^{(a_k)}(\tau)$$

with some polynomial $p_{j,k}(t)$ which consists of monomials of degree $d_{j,k}$, $a_j - 1 \ge d_{j,k} \ge a_{j-1} + 1$. Also, $\mathcal{E}(\mathbf{c})$ is obviously spanned by $\gamma^{(a_1)}(\tau), \dots, \gamma^{(a_d)}(\tau)$ so that $\frac{t^{a_d+1}}{(a_d+1)!}\mathcal{E}(\mathbf{c}) = e_1^*(\mathbf{c})\, t^{a_d+1}\gamma^{(a_1)}(\tau) + \cdots + e_d^*(\mathbf{c})\, t^{a_d+1}\gamma^{(a_d)}(\tau)$. Therefore

$$\gamma(t+\tau) = \gamma(\tau) + \sum_{k=1}^{d} \Big(\frac{t^{a_k}}{a_k!} + p_k(t) + e_k^*(\mathbf{c})t^{a_d+1}\Big)\gamma^{(a_k)}(\tau)$$

where p_k is a polynomial which consists of monomials of degree ℓ, $a_k \le \ell \le a_d$ and $\ell \notin \{a_k, a_{k+1}, \dots, a_d\}$. We set

$$t^{a_k}\varphi_k(t) = (\frac{t^{a_k}}{a_k!} + p_k(t) + e_k^*(\mathbf{c})t^{a_d+1}).$$

Then (24) follows and (25) is easy to check. This completes the proof. □

Normalization of finite type curve. Let $\mathbf{a} = (a_1, \cdots, a_d)$ be d-tuples of positive integers $a_1, a_2, \cdots, a_d$ satisfying $a_1 < a_2 < \cdots < a_d$. For $\epsilon > 0$, let us denote by $\mathfrak{G}^{\mathbf{a}}(\epsilon)$ the class of smooth curves defined in I which is given by

$$\mathfrak{G}^{\mathbf{a}}(\epsilon) = \Big\{\gamma \in C^\infty(I) : \gamma(t) = (t^{a_1}\varphi_1(t), t^{a_2}\varphi_2(t), \cdots, t^{a_d}\varphi_d(t)), \\ \Big\|\varphi_i - \frac{1}{a_i!}\Big\|_{C^{a_d+1}(I)} \le \epsilon\Big\}.$$

Let γ be of type $\mathbf{a}$ at τ. Recalling (6) and (9), for $[\tau, \tau+h]^* \subset I$ let us set

$$\gamma_\tau^{h,\mathbf{a}}(t) = [M_\tau^{\gamma,\mathbf{a}} D_h^{\mathbf{a}}]^{-1}(\gamma(ht+\tau) - \gamma(\tau)). \tag{27}$$

Here $D_h^{\mathbf{a}}$ is given by (12). Then by Lemma 3.1 it follows that

$$\gamma_\tau^{h,\mathbf{a}}(t) = (t^{a_1}\varphi_1(ht), t^{a_2}\varphi_2(ht), \dots, t^{a_d}\varphi_d(ht)) \tag{28}$$

for $\varphi_1, \dots, \varphi_d$ which are smooth functions on I and satisfy (25). Then it is easy to see the following.

Lemma 3.3. *Let $\gamma : I = [0,1] \to \mathbb{R}^d$ be a smooth curve. Suppose that γ is of type $\mathbf{a}$ at τ. Then for any $\epsilon > 0$ there is an $h_\circ = h_\circ(\mathbf{a}, \epsilon, \tau) > 0$ such that $\gamma_\tau^{h,\mathbf{a}}(t) \in \mathfrak{G}^{\mathbf{a}}(\varepsilon)$ if $[\tau, \tau+h]^* \subset I$ and $0 < |h| < h_\circ$.*

Then the curves in $\mathfrak{G}^{\mathbf{a}}(\varepsilon)$ are close to the curve

$$\gamma_\circ^{\mathbf{a}} = \Big(\frac{t^{a_1}}{a_1!}, \dots, \frac{t^{a_d}}{a_d!}\Big).$$

Similarly as in the nondegenerate case, the upper and lower bounds of the torsion of the curves can be controlled uniformly as long as the curve belongs to $\mathfrak{G}^{\mathbf{a}}(\varepsilon)$. The following is essentially same with Lemma 2 in [14].

Lemma 3.4. *Let $\gamma(t) \in \mathfrak{G}^{\mathbf{a}}(\varepsilon)$. If $\varepsilon > 0$ is sufficiently small, then there is a $B > 0$, independent of γ, such that*

$$(B/2)\, t^{\sum_{i=1}^d a_i - \frac{d(d+1)}{2}} \le \det(\gamma'(t), \cdots, \gamma^{(d)}(t)) \le 2B\, t^{\sum_{i=1}^d a_i - \frac{d(d+1)}{2}}.$$

Proof. To begin with, let us set

$$\Phi_{i,j}(t) = \sum_{k=0}^{j-1} a_i(a_i-1)\cdots(a_i-(j-k-1))\binom{j}{k} t^k \varphi_i^{(k)}(t) + t^j \varphi_i^{(j)}(t).$$

Then it is easy to see that $\frac{d^j}{dt^j}(t^{a_i}\varphi_i(t)) = t^{a_i-j}\Phi_{i,j}$. Hence, the torsion of $\gamma(t)$ can be written as

$$\begin{aligned} &\det(\gamma'(t), \cdots, \gamma^{(d)}(t)) \\ &= t^{\sum_{i=1}^d a_i - \frac{d(d+1)}{2}} \det \begin{pmatrix} \Phi_{1,1}(t) & \Phi_{1,2}(t) & \cdots & \Phi_{1,d}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{d,1}(t) & \Phi_{d,2}(t) & \cdots & \Phi_{d,d}(t) \end{pmatrix} \\ &=: t^{\sum_{i=1}^d a_i - \frac{d(d+1)}{2}} \det \Phi(t). \end{aligned}$$

Since $(t^{a_1}\varphi_1(t), t^{a_2}\varphi_2(t), \cdots, t^{a_d}\varphi_d(t)) \in \mathfrak{G}^{\mathbf{a}}(\varepsilon)$, it follows that

$$\Phi_{i,j}(t) = \Phi_{i,j}(0) + O(\epsilon) = \frac{\prod_{l=0}^{j-1}(a_i - l)}{a_i!} + O(\epsilon).$$

So, $\det \Phi(t) = \det \Phi(0) + O(\epsilon)$. Hence if ϵ is sufficiently small, $\frac{1}{2}\det \Phi(0) \le \det \Phi(t) \le 2\det \Phi(0)$. This gives the desired inequality. □

Remark 3.5. *This lemma holds for any minor of the matrix* $(\nu'(t), \nu''(t), \cdots, \nu^{(d)}(t))$. *In fact, if a* $k\times k$ *submatrix* M_k *contains* $i_1, \dots, i_k$*-th rows of* $(\nu'(t), \nu''(t), \cdots, \nu^{(d)}(t))$, *then* $\det(M_k)$ *is bounded above and below by* $t^{\sum_{l=1}^{k}(a_{i_l}-i_l)}$ *uniformly for* $\nu \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$ *if* $\epsilon > 0$ *is sufficiently small.*

Normalization via scaling. We now start the proof of Theorem (1.3). Fix $0 < \alpha \le d$ and set

$$\sigma = \frac{1}{\beta(\alpha)}\Big(\sum_{i=1}^{d} a_i - \frac{d(d+1)}{2}\Big) + 1.$$

Let γ be a finite type curve defined on I and $[\tau, \tau+h]^* \subset I$. Then let us consider the integral

$$T_\tau^h f(x) = \int_{[\tau,\tau+h]^*} e^{i\lambda x\cdot\gamma(t)} f(t) w_\gamma^\alpha(t)dt.$$

Let us set $f_\tau^h(t) = f(ht+\tau)$. Then by the change of variables $t \to ht + \tau$ and (27), it follows that

$$\begin{aligned}|T_\tau^h f(x)| &= \Big|\int_{[\tau,\tau+h]^*} e^{i\lambda x\cdot(\gamma(t)-\gamma(\tau))} f(t) w_\gamma^\alpha(t)dt\Big| \\ &= |h|\Big|\int_I e^{i\lambda D_h^{\mathbf{a}}(M_\tau^{\gamma,\mathbf{a}})^t x\cdot\gamma_\tau^{h,\mathbf{a}}(t)} f_\tau^h(t) w_\gamma^\alpha(ht+\tau)dt\Big|\end{aligned} \tag{29}$$

By (27) it follows that

$$|\det(M_\tau^{\gamma,\mathbf{a}})|^{\frac{1}{\beta(\alpha)}} |h|^{\sigma-1} w^\alpha_{\gamma_\tau^{h,\mathbf{a}}} = w_\gamma^\alpha(ht+\tau). \tag{30}$$

Hence, combining this with (29) we have

$$|T_\tau^h f(x)| = |\det(M_\tau^{\gamma,\mathbf{a}})|^{\frac{1}{\beta(\alpha)}} |h|^{\sigma} \big|T_\lambda^{\gamma_\tau^{h,\mathbf{a}}}[w^\alpha_{\gamma_\tau^{h,\mathbf{a}}}, f_\tau^h](D_h^{\mathbf{a}}(M_\tau^{\gamma,\mathbf{a}})^t x)\big|. \tag{31}$$

By Lemma 3.3 for $\tau \in I$ and $\epsilon > 0$ there are $\mathbf{a} = \mathbf{a}(\tau)$ and $h_\circ = h_\circ(\tau,\epsilon)$ such that $\gamma_\tau^{h,\mathbf{a}} \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$ provided that $[\tau, \tau+h]^* \subset I$ and $0 < |h| \le h_\circ$. Since I is compact, we can obviously decompose the interval I into finitely many intervals of disjoint interiors so that $I = \cup_{j=0}^N [\tau_j, \tau_j + h_j]^*$ and $\gamma_j = \gamma_{\tau_j}^{h_j,\mathbf{a}_j} \in \mathfrak{G}^{\mathbf{a}_j}(\epsilon)$. Then by (29) and (31) we see that

$$\begin{aligned}|T_\lambda^\gamma[w_\gamma^\alpha, f](x)| &\le \sum_{j=0}^{N} |T_{\tau_j}^{h_j} f(x)| \\ &= \sum_{j=0}^{N} |\det(M_{\tau_j}^{\gamma,\mathbf{a}_j})|^{\frac{1}{\beta(\alpha)}} |h_j|^\sigma \big|T_\lambda^{\gamma_j}[w^\alpha_{\gamma_j}, f_{\tau_j}^{h_j}](D_{h_j}^{\mathbf{a}_j}(M_{\tau_j}^{\gamma,\mathbf{a}_j})^t x)\big|.\end{aligned} \tag{32}$$

Since there are only finitely many terms, in order to show Theorem 1.3 it is enough to consider $\mu \in \mathfrak{B}(\alpha, 1)$ and $\gamma \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$ for some $\mathbf{a}$ and small $\epsilon > 0$. In fact, define a measure by $\int F(x) d\widetilde{\mu}_j = \int F((M^{\gamma,\mathbf{a}_j}_{\tau_j})^t D^{\mathbf{a}_j}_{h_j} x) d\mu$. By Lemma 2.3 $\widetilde{\mu}_j$ satisfies (4) with some constant $C_{\widetilde{\mu}_j}$ since $\det(M^{\gamma,\mathbf{a}_j}_{\tau_j}) \neq 0$. Hence if we set $\mu_j = (1 + C_{\widetilde{\mu}_j})^{-1} \widetilde{\mu}_j$, then $\mu_j \in \mathfrak{B}(\alpha, 1)$. On the other hand, from (32) we have

$$\|T^\gamma_\lambda [w^\alpha_\gamma, f]\|_{L^q(d\mu)} \le C \sum_{j=0}^{N} |\det(M^{\gamma,\mathbf{a}_j}_{\tau_j})|^{\frac{1}{\beta(\alpha)}} |h_j|^\sigma \|T^{\gamma_j}_\lambda [w^\alpha_{\gamma_j}, f^{h_j}_{\tau_j}]\|_{L^q(d\mu_j)}.$$

Suppose that (8) holds for $\mu \in \mathfrak{B}(\alpha, 1)$ and $\gamma \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$ provided that $\epsilon > 0$ is small enough. Then we have $\|T^{\gamma_j}_\lambda [w^\alpha_{\gamma_j}, f^{h_j}_{\tau_j}]\|_{L^q(d\mu_j)} \le C \lambda^{-\frac{\alpha}{q}} \|f^{h_j}_{\tau_j}\|_{L^p(w^\alpha_{\gamma_j} dt)}$. So, we get

$$\|T^\gamma_\lambda [w^\alpha_\gamma, f]\|_{L^q(d\mu)} \le \lambda^{-\frac{\alpha}{q}} \sum_{j=0}^{N} |\det(M^{\gamma,\mathbf{a}_j}_{\tau_j})|^{\frac{1}{\beta(\alpha)}} |h_j|^\sigma \|f^{h_j}_{\tau_j}\|_{L^p(w^\alpha_{\gamma_j} dt)}$$

Then by changing variables $t \to (t - \tau_j)/h_j$ and (30) it is easy to see that $\|f^{h_j}_{\tau_j}\|_{L^p(w^\alpha_{\gamma_j} dt)}$ $= |\det(M^{\gamma,\mathbf{a}_j}_{\tau_j})|^{-\frac{1}{\beta(\alpha)}} |h_j|^{-\sigma} \|f\|_{L^p(w^\alpha_\gamma dt)}$. Hence we get the desired inequality.

We are now reduced to showing that (8) holds for $\gamma \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$ and $\mu \in \mathfrak{B}(\alpha, 1)$ if $\epsilon > 0$ is sufficiently small. This will be done in what follows.

Proof of (8) *when* $\gamma \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$ *and* $\mu \in \mathfrak{B}(\alpha, 1)$. We start with breaking $T^\gamma_\lambda [w^\alpha_\gamma, f]$ dyadically so that

$$T^\gamma_\lambda [w^\alpha_\gamma, f](x) = \sum_{j=0}^{\infty} T_j f,$$

where

$$T_j f = \int_{[2^{-j-1}, 2^{-j}]} e^{i\lambda x \cdot \gamma(t)} f(t) w^\alpha_\gamma(t) dt.$$

In order to prove (8) it is sufficient to show that

$$\|T_j f\|_{L^q(d\mu)} \le C 2^{-j\sigma(1 - \frac{\beta(\alpha)}{q} - \frac{1}{p})} \lambda^{-\frac{\alpha}{q}} \|f\|_{L^p(w^\alpha_\gamma dt)}. \tag{33}$$

Let us set

$$\int F(x) d\mu_j(x) = \frac{2^{-j\beta(\alpha)\sigma}}{1 + C\|(M^{\gamma,\mathbf{a}}_0)^{-t}\|^\alpha} \int F(D^{\mathbf{a}}_{2^{-j}} (M^{\gamma,\mathbf{a}}_0)^t x) d\mu(x).$$

By rescaling as before (cf. (29)) it follows that

$$\|T_j f\|_{L^q(d\mu)} \le C 2^{-j\sigma(1 - \frac{\beta(\alpha)}{q})} \|\mathcal{T}_j f_j\|_{L^q(d\mu_j)} \tag{34}$$

where $f_j(t) = f(2^{-j} t)$ and

$$\mathcal{T}_j g = \int_{[\frac{1}{2}, 1]} e^{i\lambda x \cdot \gamma^{2^{-j}, \mathbf{a}}_0(t)} g(t) [2^{(\sigma-1)j} w^\alpha_\gamma(2^{-j} t)] dt.$$

Then by Lemma 2.3 $\mu_j \in \mathcal{B}(\alpha, 1)$, and by rescaling it is easy to see that $\gamma^{2^{-j}, \mathbf{a}}_0 \in \mathfrak{G}^{\mathbf{a}}(C 2^{-j} \epsilon)$. If $\epsilon > 0$ is small enough, by Lemma 3.4 it follows that $B_1 t^{\sigma-1} \le w^\alpha_\gamma(t) \le$

$B_2 t^{\sigma-1}$, $t \in I$ with B_1, B_2, independent of $\gamma \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$. Hence if $\epsilon > 0$ is sufficiently small, then

$$w_\gamma^\alpha(2^{-j}t) \sim 2^{-(\sigma-1)j} \sim 2^{-(\sigma-1)j} w_\gamma^\alpha(t),\ t \in [1/2, 1]$$

with the implicit constant independent of γ as long as $\gamma \in \mathfrak{G}^{\mathbf{a}}(C2^{-j}\epsilon)$. Hence we may disregard the weight. So, for (34) it is enough to show uniform estimate $\|\mathcal{T}_j g\|_{L^q(d\mu_j)} \le C\|g\|_{L^p}$ for all $j \ge 0$. Therefore we are reduced to showing that if $\gamma \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$ and $\mu \in \mathcal{B}(\alpha, 1)$, there is a uniform constant C such that

$$\|T_*^\gamma f\|_{L^q(d\mu)} \le C\lambda^{-\frac{\alpha}{q}} \|f\|_{L^p}, \tag{35}$$

where

$$T_*^\gamma f(x) = \int_{[\frac12, 1]} e^{i\lambda x\cdot\gamma(t)} f(t)dt.$$

Obviously the curve $\gamma \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$ is non-degenerate uniformly on $[\frac12, 1]$. More precisely let $\gamma \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$, $[\tau, \tau + h] \subset [\frac12, 1]$ and consider the curve γ_τ^h which is given by

$$\gamma_\tau^h(t) = [M_\tau^\gamma D_h]^{-1}(\gamma(ht + \tau) - \gamma(\tau)).$$

Since $\tau \in [\frac12, 1]$ and $\gamma \in \mathfrak{G}^{\mathbf{a}}(\epsilon)$, it follows that $\|(M_\tau^\gamma)^{-1}\| \le C$ uniformly if $\epsilon > 0$ is small enough. Hence following the argument in the proof Lemma 2.1 it is easy to see that there is an h_0, independent of γ, such that $\gamma_\tau^h \in \mathfrak{G}(\epsilon)$ if $h \le h_0$ and $[\tau, \tau + h] \subset [\frac12, 1]$ (see remark 2.2). Hence, we may proceed similarly with the lines of argument in the first part of *Proof Theorem 1.1.* Breaking the interval $[\frac12, 1]$ into $O(1/h_0)$ essentially disjoint intervals, by normalization via translation and rescaling we see that $\|T_*^\gamma f\|_{L^q(d\mu)}$ is bounded by a sum of as many as $O(1/h_0)$ of $C\|T_\lambda^{\widetilde{\gamma}} f\|_{L^q(d\widetilde{\mu})}$ while $\widetilde{\gamma} \in \mathfrak{G}(\epsilon)$ and $\widetilde{\mu} \in \mathfrak{B}(\alpha, 1)$ (cf. (23)). Finally, from Remark 2.9 we see that if $\epsilon > 0$ is sufficiently small there is a uniform constant, independent of $\widetilde{\gamma}$ and $\widetilde{\mu}$, such that $\|T_*^\gamma f\|_{L^q(d\mu)} \le C\lambda^{-\frac{\alpha}{q}}\|f\|_p$ whenever $\widetilde{\gamma} \in \mathfrak{G}(\epsilon)$ and $\widetilde{\mu} \in \mathfrak{B}(\alpha, 1)$. Therefore we get (35). This completes the proof.

Remark 3.6. *Since we only rely on scaling and stability of the estimates for non-degenerate case, the argument here also works for monomial type curves which were considered in* [14]. *In fact, let $0 < a_1 < \cdots < a_d$ be real numbers and suppose that $\gamma(t) = (t^{a_1}\varphi_1(t), \dots, t^{a_d}\varphi_d(t))$, $\varphi_i(0) \ne 0$ and $\lim_{t\to 0} t^k \varphi_i^{(k)}(t) = 0$ for $k = 1, \dots, d$. Then if $d/q \le (1-1/p)$, $q \ge 2d$, $\beta(\alpha)/q + 1/p < 1$ and $q > \beta(\alpha) + 1$, for a sufficiently small $\delta > 0$ the following estimate holds;*

$$\Big\| \int_0^\delta e^{i\lambda x\cdot\gamma(t)} f(t) w_\gamma^\alpha(t)dt \Big\|_{L^q(d\mu)} \le C\|f\|_{L^p(w_\gamma^\alpha dt)}.$$

Appendix A. A necessary condition for the estimates (5) and (8)

We show that (5) and (8) hold only if

$$\beta(\alpha)/q + 1/p \le 1. \tag{36}$$

It is sufficient to consider (8) since (5) is a special case of (8). To see this let us fix j so that $d-j-1<\alpha\le d-j$. We consider a measure μ which is defined by

$$d\mu(x)=\prod_{i=1}^{j} d\delta(x_i)\,|x_{j+1}|^{\alpha-d+j}\,dx_{j+1}dx_{j+2}\cdots dx_d.$$

Here δ is the delta measure. Then it follows that $\int_{B(x,\rho)} d\mu(x)\le C\rho^{\alpha-d+j+1}\cdot\rho^{d-j-1}=C\rho^{\alpha}$, i.e. (4) is satisfied. Now let $\gamma(t)$ be a curve of finite type $\mathbf{a}$ at τ. So, $M_\tau^{\gamma,\mathbf{a}}$ is nonsingular. We choose h small enough so that $\gamma_\tau^{h,\mathbf{a}}\in\mathfrak{G}^{\mathbf{a}}(\epsilon)$ for a small ϵ. We define a measure $\widetilde{\theta}$ by

$$\int F(x)d\widetilde{\theta}(x)=\int F((M_1^{\gamma_\circ^{\mathbf{a}}})^{-t}(M_\tau^{\gamma,\mathbf{a}})^{-t}(D_h^{\mathbf{a}})^{-1}x)d\mu(x).$$

It is easy to show that $d\widetilde{\theta}$ also satisfies (4).

By taking $f(t)=\chi_{[\tau+h-h\lambda^{-\frac1d},\tau+h]}(t)$ (see (29)) and making changes of variables $t\to ht+\tau$ we have $|T_\lambda^\gamma[w_\gamma^\alpha,f](x)|=\Big|h\int_{1-\lambda^{-\frac1d}}^1 e^{i\lambda D_h^{\mathbf{a}}(M_\tau^{\gamma,\mathbf{a}})^t x\cdot\gamma_\tau^{h,\mathbf{a}}(t)}w_\gamma^\alpha(ht+\tau)dt\Big|$. Then it follows that

$$\|T_\lambda^\gamma[w_\gamma^\alpha,f]\|_{L^q(d\widetilde{\theta})}^q=h^q\int\Big|\int_{1-\lambda^{-\frac1d}}^1 e^{i\lambda x\cdot(M_1^{\gamma_\circ^{\mathbf{a}}})^{-1}\gamma_\tau^{h,\mathbf{a}}(t)}w_\gamma^\alpha(ht+\tau)dt\Big|^q d\mu(x)$$
$$=h^q\int\Big|\int_{-\lambda^{-\frac1d}}^0 e^{i\lambda x\cdot(M_1^{\gamma_\circ^{\mathbf{a}}})^{-1}[\gamma_\tau^{h,\mathbf{a}}(t+1)-\gamma_\tau^{h,\mathbf{a}}(1)]}w_\gamma^\alpha(ht+h+\tau)dt\Big|^q d\mu(x).$$

By (30) and Lemma 3.4, $w_\gamma^\alpha(ht+\tau)=|\det(M_\tau^{\gamma,\mathbf{a}})|^{\frac{1}{\beta(\alpha)}}h^{\sigma-1}w^\alpha_{\gamma_\tau^{h,\mathbf{a}}}(t)\sim h^{\sigma-1}|t|^{\sigma-1}$. Note that $\gamma_\tau^{h,\mathbf{a}}$ is nondegenerate on the interval $[\frac12,1]$ since $\gamma_\tau^{h,\mathbf{a}}$ is close to $\gamma_\circ^{\mathbf{a}}$ by (28). By Taylor's expansion (cf. Lemma 2.1), $(M_1^{\gamma_\circ^{\mathbf{a}}})^{-1}[\gamma_\tau^{h,\mathbf{a}}(t+1)-\gamma_\tau^{h,\mathbf{a}}(1)]=\gamma_\circ(t)+O(t^{d+1})$. Hence it is easy to see that

$$\Big|\int_{-\lambda^{-\frac1d}}^0 e^{i\lambda x\cdot(M_1^{\gamma_\circ^{\mathbf{a}}})^{-1}[\gamma_\tau^{h,\mathbf{a}}(t+1)-\gamma_\tau^{h,\mathbf{a}}(1)]}w_\gamma^\alpha(ht+h+\tau)dt\Big|\gtrsim h^{\sigma-1}\lambda^{-\frac1d}$$

if $x\in\mathcal{R}=\{x=(x_1,\dots,x_d):|x_i|\le c\lambda^{\frac id-1}\}$ with a small $c>0$. Also note that $\|f\|_{L^p(w_\gamma^\alpha dt)}\lesssim\lambda^{-\frac{1}{dp}}$. Hence (8) implies

$$\lambda^{-\frac{\alpha}{q}-\frac{1}{dp}}\gtrsim\lambda^{-\frac1d}\big(\theta(\mathcal{R})\big)^{\frac1q}.$$

By a computation $\theta(\mathcal{R})\sim\lambda^{-\alpha+\frac{\beta(\alpha)}{d}}$. Hence, $\lambda^{-\frac{\alpha}{q}-\frac{1}{dp}}\gtrsim\lambda^{-\frac1d}\lambda^{-\frac{\alpha}{q}+\frac{\beta(\alpha)}{dq}}$. Letting $\lambda\to\infty$ gives the condition (36).

Appendix B. Proof of Lemma 2.4.

Here we provide a proof of Lemma 2.4. For $1\le n\le d$ let us set

$$E_n=\{\mathbf{t}\in I^n:0<t_1<\cdots<t_n\}.$$

We need to show that $\Gamma_\gamma:E_d\to\mathbb{R}^d$ is one-to-one provided that $\gamma\in\mathfrak{G}^{\mathbf{a}}(\epsilon)$ and $\epsilon>0$ is sufficiently small. Since $\gamma\in\mathfrak{G}^{\mathbf{a}}(\epsilon)$, it is obvious that the determinant of $\frac{\partial\Gamma_\gamma(\mathbf{t})}{\partial\mathbf{t}}$

and all its minors take the form $\det(q'(t_{\alpha_1}), \dots, q'(t_{\alpha_n}))$ while $\alpha_1, \dots, \alpha_n \in \{1, \dots, d\}$ and $q \in \mathfrak{G}^{\mathbf{b}}(\epsilon)$ for some $\mathbf{b} = (b_1, \dots, b_n)$, $b_1 < \dots < b_n$, $b_1, \dots, b_n \in \{a_1, \dots, a_d\}$. Here $\mathfrak{G}^{\mathbf{b}}(\epsilon)$ and $\gamma_\circ^{\mathbf{b}}$ is defined similarly as before. Hence, by the argument in [19] (also see [15, Section 6]) which is originally due to Steinig [26], we only need to show that $\det(q'(t_{\alpha_1}), \dots, q'(t_{\alpha_n}))$ is single signed and nonzero for $(t_{\alpha_1}, \dots, t_{\alpha_n}) \in E_n$ provided that $q \in \mathfrak{G}^{\mathbf{b}}(\epsilon)$ and $\epsilon > 0$ is sufficiently small. Therefore the following lemma completes the proof.

Lemma B.1. *For $1 \le n \le d$, let $\mathbf{b} = (b_1, \cdots, b_n)$ and $b_1, b_2, \dots, b_n$ be positive integers satisfying that $b_1 < b_2 < \dots < b_n$. Let $\gamma \in \mathfrak{G}^{\mathbf{b}}(\epsilon)$ and set $\Gamma_\gamma(\mathbf{t}) = \sum_{i=1}^n \gamma(t_i)$, $\mathbf{t} = (t_1, \dots, t_n) \in I^n$. Then if $\epsilon = \epsilon(\mathbf{b}, n) > 0$ is sufficiently small, there is a constant C, independent of γ, such that if $\mathbf{t} = (t_1, \dots, t_n) \in E_n$,*

$$\det\left(\frac{\partial \Gamma_\gamma(\mathbf{t})}{\partial \mathbf{t}}\right) \ge C \prod_{i=1}^n \left| \det\Big((\gamma_\circ^{\mathbf{b}})'(t_i), \cdots, (\gamma_\circ^{\mathbf{b}})^{(n)}(t_i)\Big)\right|^{\frac{1}{n}} \prod_{1 \le i < j \le n} (t_j - t_i). \tag{37}$$

Proof. We shall be brief since the proof here is an adaptation of the argument in [14]. Let $\Phi_k(t)$ be a $k \times k$ minor of $\Phi_n(t) := \det \Phi(t)$, which consists of $\Phi_{i,j}(t)$ with $1 \le i, j \le k$.(See Lemma 3.4.)

Adopting the notations in [15, 14], we define a sequence of functions I_k, $1 \le k \le n$ as follows:

$$I_1(t) = \frac{t^{\sum_{i=1}^{n-2}(a_i - i)} \Phi_{n-2}(t)\, t^{\sum_{i=1}^{n}(a_i - i)} \Phi_n(t)}{(t^{\sum_{i=1}^{n-1}(a_i - i)} \Phi_{n-1}(t))^2} = t^{a_n - a_{n-1} - 1} \frac{\Phi_{n-2}(t)\Phi_n(t)}{\Phi_{n-1}(t)^2}$$

and

$$\begin{aligned} I_k(t_1, \dots, t_k) = \prod_{l=1}^k t_l^{a_{n-k+1} - a_{n-k} - 1} \frac{\Phi_{n-k-1}(t_l)\Phi_{n-k+1}(t_l)}{\Phi_{n-k}(t_l)^2} \\ \times \int_{t_1}^{t_2} \cdots \int_{t_{k-1}}^{t_k} I_{k-1}(s_1, \dots, s_{k-1}) ds_{k-1} \cdots ds_1 \end{aligned}$$

with $\Phi_{-1}, \Phi_0 \equiv 1$. By Lemma 3.4 and Remark 3.5, there are positive constants G_k, uniform in $\gamma \in \mathfrak{G}^{\mathbf{b}}(\epsilon)$, such that $\frac{1}{2} G_k \le \Phi_k(t) \le 2G_k$ for all $t \in I$ and sufficiently small $\varepsilon > 0$. Hence $I_1(t) \gtrsim t^{a_n - a_{n-1} - 1} G_{n-2}\, G_n / G_{n-1}^2$.

Now we claim that for $1 \le k \le n-2$,

$$I_k(t_i, \dots, t_k) \gtrsim \frac{G_{n-k-1}^k\, G_n}{G_{n-k}^{k+1}} \prod_{l=1}^k t_l^{\frac{1}{k}\sum_{i=n-k+1}^n (a_i - i) - (a_{n-k} - (n-k))} \prod_{1 \le i < j \le k} (t_j - t_i) \tag{38}$$

also holds uniformly in $\gamma \in \mathfrak{G}^{\mathbf{b}}(\epsilon)$. Suppose that (38) holds for $k \le n-3$. Then, it follows that

$$I_{k+1}(t_1, \dots, t_{k+1}) \gtrsim \left(\frac{G_{n-k-2}G_{n-k}}{G_{n-k-1}^2}\right)^{k+1} \frac{G_{n-k-1}^k\, G_n}{G_{n-k}^{k+1}} \prod_{l=1}^{k+1} t_l^{a_{n-k}-a_{n-k-1}-1}$$
$$\times \int_{t_1}^{t_2} \cdots \int_{t_k}^{t_{k+1}} \prod_{l=1}^{k} s_l^{(\frac{1}{k}\sum_{i=n-k+1}^{n}(a_i-i)-(a_{n-k}-(n-k)))} \prod_{1\le i<j\le k}(t_j-t_i)\,ds_k\cdots ds_1$$
$$\gtrsim \frac{G_{n-k-2}^{k+1} G_n}{G_{n-k-1}^{k+2}} \prod_{l=1}^{k+1} t_l^{(\frac{1}{k+1}\sum_{i=n-k}^{n}(a_i-i)-(a_{n-k-1}-(n-k-1)))} \prod_{1\le i<j\le k+1}(t_j-t_i).$$

The first inequality is valid uniformly in γ whenever $\gamma \in \mathfrak{G}^{\mathbf{b}}(\varepsilon_0)$, and the last inequality is established by modifying Corollary 7 in [14], which is irrelevant to the structure of γ. The remaining cases $k = n-1, n$ can also be handled similarly by making use of (38) successively. So, it follows that

$$I_n(t_1, \dots, t_n) \gtrsim G_n \prod_{l=1}^{n} t_l^{\frac{1}{n}\sum_{i=1}^{n}(a_i-i)} \prod_{1\le i<j\le n}(t_j-t_i)$$

holds uniformly. Since $I_n(t_1, \dots, t_n) = \det \partial\Gamma_\gamma(\mathbf{t})/\partial\mathbf{t}$ (see Section 5 in [15]) and $t_l^{\sum_{i=1}^{n}(a_i-i)} \sim |\det((\gamma_\circ^{\mathbf{b}})'(t_l), \dots, (\gamma_\circ^{\mathbf{b}})^{(n)}(t_l))|$, we conclude that (37) holds uniformly for $\gamma \in \mathfrak{G}^{\mathbf{b}}(\varepsilon_0)$. $\square$

References

[1] G.I. Arkhipov, V.N. Chubarikov and A.A. Karatsuba, Trigonometric sums in number theory and analysis, Translated from the 1987 Russian original. de Gruyter Expositions in Mathematics, 39, Berlin, 2004.

[2] J. Bennett, A. Carbery and T. Tao, On the multilinear restriction and Kakeya conjectures, *Acta. Math.*, **196** (2006), 261–302.

[3] J.-G. Bak and S. Lee, Estimates for an oscillatory integral operator related to restriction to space curves, *Proc. Amer. Math. Soc.*, **132** (2004), 1393–1401.

[4] J. Bennett and A. Seeger, The Fourier extension operator on large spheres and related oscillatory integrals, *Proc. London. Math. Soc.*, **98** (2009), 45–82.

[5] J.-G. Bak and D. Oberlin, A note on Fourier restriction for curves in $\mathbb{R}^3$, Proceedings of the AMS Conference on Harmonic Analysis, Mt. Holyoke College (June 2001), Contemp. Math., Vol. 320, Amer. Math. Soc., Providence, RI, 2003.

[6] J.-G. Bak, D. Oberlin and A. Seeger, Restriction of Fourier transforms to curves, II: Some classes with vanishing torsion, *J. Austr. Math. Soc.*, **85** (2008), 1–28.

[7] ———, Restriction of Fourier transforms to curves and related oscillatory integrals, *Amer. J. Math.*, **131** (2009), 277–311.

[8] ———, Restriction of Fourier transforms to curves: An endpoint estimate with affine arclength measure, *to appear in J. Reine Angew. Math.*

[9] J. Bennett, A. Carbery, F. Soria and A. Vargas, A Stein conjecture for the circle, *Math. Annalen.*, **336** (2006), 671–695.

[10] J. Bourgain and L. Guth, Bounds on oscillatory integral operators based on multilinear estimates, *Geom. Funct. Anal.*, **21** (2011), 1239–1295.

[11] M. Christ, On the restriction of the Fourier transform to curves: endpoint results and the degenerate case, *Trans. Amer. Math. Soc.*, **287** (1985), 223–238.

[12] S. Dendrinos, M. Folch-Gabayet and J. Wright, An affine-invariant inequality for rational functions and applications in harmonic analysis, *Proc. Edinb. Math. Soc.*, **53** (2010), 639–655.

[13] S. Dendrinos, N. Laghi and J. Wright, Universal L^p improving for averages along polynomial curves in low dimensions, *J. Funct. Anal.*, **257** (2009), 1355–1378.

[14] S. Dendrinos and D. Müller, Uniform estimates for the local restriction of the Fourier transform to curves, *to appear in Trans. Amer. Math. Soc.*

[15] S. Dendrinos and J. Wright, Fourier restriction to polynomial curves I: A geometric inequality, *Amer. J. Math.*, **132** (2010), 1031–1076.

[16] S. W. Drury, Restriction of Fourier transforms to curves, *Ann. Inst. Fourier.*, **35** (1985), 117–123.

[17] ———, Degenerate curves and harmonic analysis, *Math. Proc. Cambridge Philos. Soc.*, **108** (1990), 89–96.

[18] S.W. Drury and B. Marshall, Fourier restriction theorems for curves with affine and Euclidean arclengths, *Math. Proc. Cambridge Philos. Soc.*, **97** (1985), 111–125.

[19] ———, Fourier restriction theorems for degenerate curves, *Math. Proc. Cambridge Philos. Soc.*, **101** (1987), 541–553.

[20] C. Fefferman, Inequalities for strongly singular convolution operators, *Acta. Math.*, **124** (1970), 9–36.

[21] A. Greenleaf and A. Seeger, On oscillatory integral operators with folding canonical relations, *Studia Math.*, **132** (1999), 125–139.

[22] L. Hörmander, Oscillatory integrals and multipliers on FL^p, *Ark. Mat.*, **11** (1973), 1–11.

[23] D. Oberlin, Fourier restriction estimates for affine arclength measures in the plane, *Proc. Amer. Math. Soc.*, **129** (2001), 3303–3305.

[24] E. Prestini, Restriction theorems for the Fourier transform to come manifolds in $\mathbb{R}^n$, *Proc. Sympos. Pure. Math.*, **35** (1979), 101–109.

[25] P. Sjölin, Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in R^2, *Studia Math.*, **51** (1974), 169–182.

[26] J. Steinig, On some rules of Laguerre's, and systems of equal sums of like powers, *Rend. Mat.*, (6) 4 (1971), 629–644 (1972).

[27] T. Tao, A. Vargas and L. Vega, A bilinear approach to the restriction and Kakeya conjectures, *J. Amer. Math. Soc.*, **11** (1998), 967–1000.

[28] A.Zygmund, On Fourier coefficients and transforms of functions of two variables, *Studia Math.*, **50** (1974), 189–201.

Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Republic of Korea

E-mail address: `hamsh@snu.ac.kr`

E-mail address: `shklee@snu.ac.kr`